APPLICATION NOTE 30

DFT or FFT? A Comparison of Fourier Transform Techniques

This application note investigates differences in performance between the DFT (Discrete Fourier Transform) and the FFT(Fast Fourier Transform) from a mathematical and practical level when utilised for harmonic analysis of electrical waveforms. Exploration of the advantages and disadvantages of each algorithm, as well as determination of the best approach for harmonic analysis within a power electronics environment will be covered. With many Power Analyzers utilising the FFT and fewer utilising the DFT, the application note will investigate the reasons for the popularity of the FFT and the problems associated with it.

Definition

Fourier analysis is a mathematical method of representing a periodic waveform in terms of a series of trigonometric functions. The French mathematician and Physicist Mr Jean Baptiste Joseph, Baron De Fourier developed the Fourier analysis technique in the 18th~19th centuries.

Fourier analysis is used in many industries, electronics, mechanical engineering, acoustic engineering and much more. This application note will apply Fourier analysis to the decomposition of AC periodic waveforms. This is commonly used within electronics to determine the distortion of a waveform by representing a time varying periodic signal as a discrete number of "harmonics".

Periodic Functions

A periodic function should be considered as a "repeating pattern", where at time nT, for all integer values of n, and T equal to the repetition period the waveform, the following is true;

$$f(x + nT) = f(x) \quad (1)$$

For example :

$y = \sin(x)$is periodic in x with a period of 2π, if $y = \sin(\omega t)$ then the period of the waveform is $\frac{2\pi}{\omega}$ where $\omega = periodic\ frequency\ (2\pi f)$

For a frequency of 10Hz, $\omega = 20\pi \frac{rads}{sec}$ therefore the fundamental time period T is $\frac{2\pi}{20\pi} = 0.1 seconds$.

$$y = \sin\big(\omega(t + nT)\big) \quad (2)$$

The magnitude of the waveform at initial time $t = 0.021$ is calculated, as well as the subsequent magnitudes at time $t + (n * T)$

$$\sin\big(20\pi(0.021 + n * 0.1)\big) \quad (3)$$

n=0	n=1	n=2
0.969	0.969	0.969

A sine wave is known as a continuous function, it does not exhibit any sudden jumps or breaks. Whereas other waveforms, such as the square wave exhibits finite discontinuities. An advantage of the Fourier series is that it can be applied to both continuous and discontinuous waveforms.

Fourier Series

The fundamental basis of the Fourier series is that a function defined within the interval $-\pi \leq x \leq \pi$ can be expressed as a convergent trigonometric series in the following form;

$$f(x) = a_0 + a_1 cosx + a_2 cos2x + a_3 cos3x + \cdots + b_1 sinx + b_2 sin2x + b_3 sin3x + \cdots$$

$$where\ a_0, a_1, a_2, a_3 \ldots\ b_1, b_2, b_3 \ldots are\ real\ constants$$

This can be simplified to the following;

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad (4)$$

For the range $-\pi\ to\ \pi$;

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \quad (5)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos\ nx\ dx \quad (n = 1, 2, 3 \ldots) \quad (6)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin\ nx\ dx \quad (n = 1, 2, 3 \ldots) \quad (7)$$

An alternative way of writing the Fourier series above, which is of more relevance to digital signal processing of complex electrical waveforms is derived from the following relationship;

$$a \cos x + b \cos x = c \sin(x + \alpha) \quad (8)$$

Applying this to the Fourier series;

$$f(x) = a_0 + c_1 \sin(x + \alpha) + c_2 \sin(2x + \alpha_2) + \ldots + c_n \sin(3x + \alpha_n) \quad (9)$$

$$a_0 is\ a\ constant$$

$c_1 = \sqrt{a_1^2 + b_1^2}, \ldots. c_n = \sqrt{a_n^2 + b_n^2}$ are the magnitudes of each coefficient (10)

$\alpha_n = \arctan\left(\frac{a_n}{b_n}\right)$ is the phase offset of each coefficient (11)

C_n represents the magnitude of the nth component, or nth harmonic. When n = 1, the harmonic is known as the fundamental.

α_n represents the phase angle of the nth harmonic.

It is at this point that all of the mathematical tools required to decompose a periodic waveform have been derived.

Mathematical Example, 2Vpk Square Wave

In this example a 2Vpk (4Vpk-pk) square wave is decomposed into its constituent harmonics.

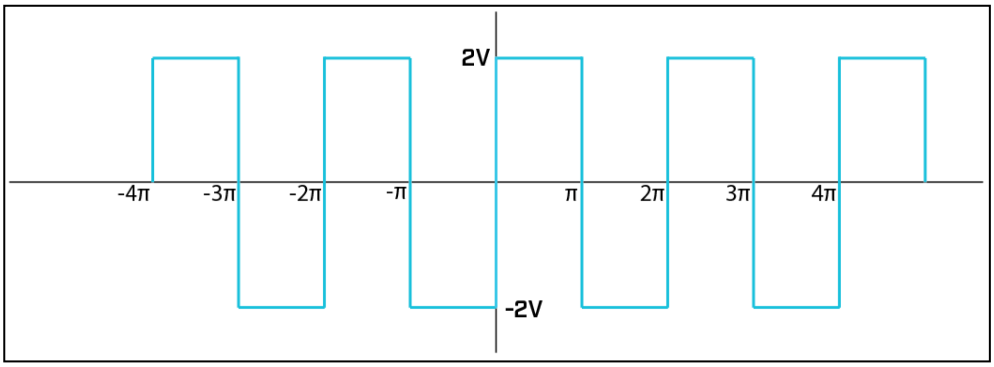

Fig 1.

The function for the above waveform is given by;

$$f(x) = \begin{cases} -2, & when -\pi < x < 0 \\ 2, & when\ 0 < x < \pi \end{cases}$$

As the pure square wave function is discontinuous, the integration will be performed in two discrete halves. One from -π to 0 and another from 0 to π.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \tag{12}$$

$$a_0 = \frac{1}{2\pi}[\int_{-\pi}^{0} -2\,dx + \int_{0}^{\pi} 2\,dx] \tag{13}$$

$$a_0 = \frac{1}{2\pi}\{[-2x]_{-\pi}^{0} + [2x]_{0}^{\pi}\} = 0 \tag{14}$$

a_0 equates to the mean value of the waveform (DC offset in electrical terms) over the complete fundamental period.

$$a_n = \frac{1}{\pi}[\int_{-\pi}^{0} -2\cos\ nx\,dx + \int_{0}^{\pi} 2\cos\ nx\,dx\,] \tag{15}$$

$$a_n = \frac{1}{\pi}\left[-\frac{2\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{2\sin nx}{n}\right]_{0}^{\pi} = 0 \tag{16}$$

Therefore, $a_1, a_2, a_3 \ldots are\ all\ zero$.

$$b_n = \frac{1}{\pi}[\int_{-\pi}^{0} -2\sin\ nx\,dx + \int_{0}^{\pi} 2\,sin\,nx\,dx\,] \tag{17}$$

$$b_n = \frac{1}{\pi}\left[\frac{2\cos nx}{n}\right]_{-\pi}^{0} + \left[\frac{-2\cos nx}{n}\right]_{0}^{\pi} = 0 \tag{18}$$

when n is odd

$$b_n = \frac{2}{\pi}\left\{\left[\left(\frac{1}{n}\right) - \left(-\frac{1}{n}\right)\right] + \left[-\left(-\frac{1}{n}\right) - \left(-\frac{1}{n}\right)\right]\right\} \tag{19}$$

$$b_n = \frac{k}{\pi}\left\{\frac{2}{n} + \frac{2}{n}\right\} = \frac{4k}{n\pi} \quad (20)$$

$$as\ k = 2, \quad b_n = \frac{8}{n\pi} \quad (21)$$

Therefore $b_1 = \frac{8}{\pi}$, $b_3 = \frac{8}{3\pi}$, $b_5 = \frac{8}{5\pi}$

when n is even

$$b_n = \frac{2}{\pi}\left\{\left[\left(\frac{1}{n}\right) - \left(\frac{1}{n}\right)\right] + \left[-\left(\frac{1}{n}\right) - \left(-\frac{1}{n}\right)\right]\right\} = 0 \quad (22)$$

$$b_1, b_2, b_3 \ldots = 0$$

Therefore, the Fourier series for the 2Vpk square wave is as follows;

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad (23)$$

$$f(x) = 0 + \sum_{n=1}^{\infty}(0 + b_n \sin nx) \quad (24)$$

$$f(x) = \frac{8}{\pi} sinx + \frac{8}{3\pi} sin3x + \frac{8}{5\pi} sin5x + \cdots \quad (25)$$

$$f(x) = \frac{8}{\pi}\left(sinx + \frac{1}{3} sin3x + \frac{1}{5} sin5x + \cdots\right) \quad (26)$$

The magnitude and phase of each harmonic can be computed as follows;

$c_n = \sqrt{a_n^2 + b_n^2}$ are the magnitudes of each coefficient (27)

$\alpha_n = \arctan\left(\frac{a_n}{b_n}\right)$ is the phase offset of each coefficient (28)

For the 3rd harmonic;

$$c_3 = \sqrt{0^2 + \frac{8}{3\pi}^2} = \frac{8}{3\pi} \quad (29)$$

$$\alpha_n = \arctan\left(\frac{0}{\left(\frac{8}{3\pi}\right)}\right) = 0° \quad (30)$$

3rd Harmonic = 0.85V, 0°

Harmonic Analysis

It is widely known that many of the waveforms within the electronics industry can be represented by simple mathematical expressions. By utilising the Fourier series, the magnitude and phase of the harmonics can be derived. Some waveforms do not fit into this category and analysis of such waveforms is performed through numerical methods. Harmonic Analysis is the process of resolving a periodic, non-sinusoidal waveform into a series of sinusoids of increasing order of frequency.

Following on from the mathematical expressions derived earlier in this application note, a numerical method is now presented which introduces the fundamentals behind the approach used within harmonic analyzers.

Ultimately, Fourier coefficients a_0, a_n and b_n will need to be determined - see equations (5), (6) and (7), this will require integration.

Numerically, the integral functions (5), (6) and (7) can be described as mean values, as follows;

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{0}^{2\pi} f(x)dx \qquad (5)$$

$$a_0 = mean\ value\ of\ f(x) \qquad (31)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\ nx\ dx = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos\ nx\ dx \quad (n = 1, 2, 3 \ldots) \qquad (6)$$

$$a_n = twice\ mean\ value\ of\ f(x)\cos nx \qquad (32)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\ nx\ dx \quad (n = 1, 2, 3 \ldots) \qquad (7)$$

$$b_n = twice\ mean\ value\ of\ f(x)\sin nx \qquad (33)$$

This integration is performed within the signal processing elements of a harmonic analyzer (traditionally a DSP) with the ***trapezoidal rule***. As most waveforms within the realm of electronics are periodic, the period of the waveform can be referred to as *w (window)*, which can be divided up into *s (sample points) of* equal parts. Where *d(time delta)* is the time interval in between the sample points.

A complex waveform is now used as an example (fig.2) to illustrate how the waveform is sampled and numerical Fourier analysis methods are explained.

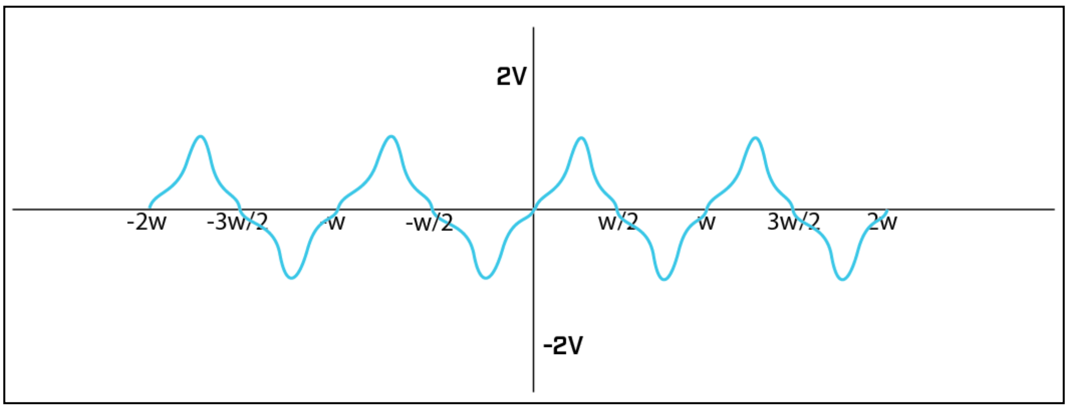

Fig 2.

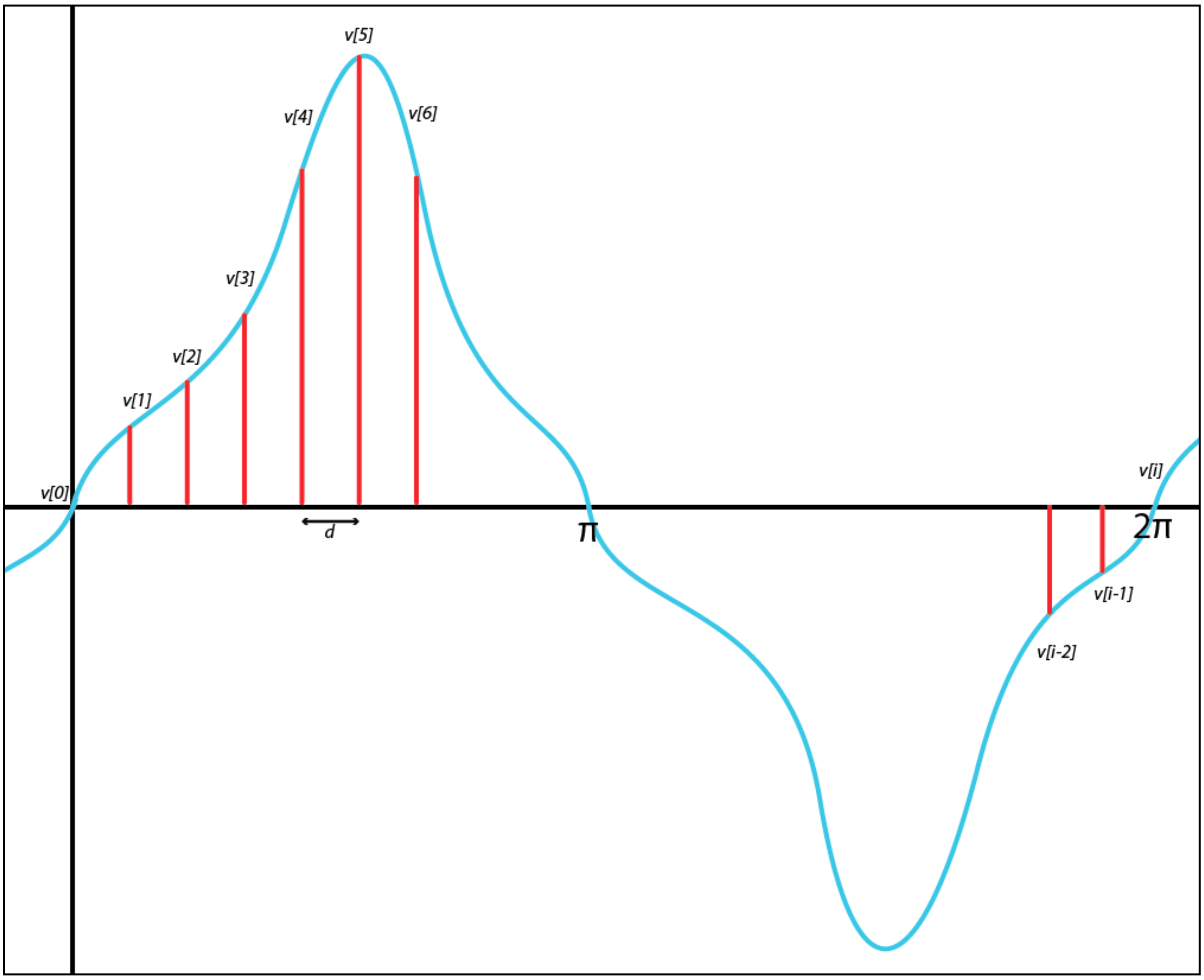

Fig 3.

$$n = total\ number\ of\ samples, i^{th} sample, d = time\ between\ samples$$

$$a_0 = Mean\ Value = \frac{area}{length\ of\ base} = \frac{1}{2\pi} * \left(\frac{2\pi}{n}\right)\sum_{i=1}^{n} v[i] = \frac{1}{n}\sum_{i=1}^{n} v[i] \tag{34}$$

$$a_n = \frac{1}{n}\sum_{i=1}^{n} v[i]\cos\left(\frac{2\pi hi}{n}\right) \tag{35}$$

$$b_n = \frac{1}{n}\sum_{i=1}^{n} v[i]\sin\left(\frac{2\pi hi}{n}\right) \tag{36}$$

In electrical terms, a_0 is the DC offset, a_n is the quadrature component of the nth harmonic, and b_n is the in-phase component of the n^{th} harmonic.

Simple trigonometry can be applied to derive the magnitude and phase of the harmonics.

As equations 35 and 36 represent the mean value of the harmonics, it is common to refer to the equivalent RMS values, a scale factor of $\sqrt{2}$ is applied to achieve this.

$$a_n = \frac{\sqrt{2}}{n}\sum_{i=1}^{n} v[i]\cos\left(\frac{2\pi hi}{n}\right) \tag{37}$$

$$b_n = \frac{\sqrt{2}}{n}\sum_{i=1}^{n} v[i]\sin\left(\frac{2\pi hi}{n}\right) \tag{38}$$

Harmonic analyzers make use of the trapezoidal rule to closely approximate the true value of the harmonic magnitudes. It is logical to assume that a higher sample rate results better accuracy and although this is the case in simplistic terms, the reality is more complicated as averaging reduces the effect of lower sample rates. That being said, as per Nyquist's theorem, the sampling rate must be twice that of the bandwidth of the measured signal.

FFT Application

All previous content within this application note represents the DFT, this could be considered the purest of Fourier transforms and is the original Fourier transform. In 1965 Cooley and Tukey published the generic FFT algorithm and the drive for this development was not improved accuracy but it was improved computation time.
Whilst the FFT required less processing power, in a real world measurement environment it is difficult for the FFT to provide the same accuracy as the DFT. The reasons for this are due to the fact that the FFT is restricted (by the nature of its "divide and conquer" mathematical approach) to a window size of 2^n samples. If the sample rate were infinitely variable, and the fundamental frequency of the waveform to be analysed already known - then it would be possible to vary the sample rate in order to ensure an integer number of cycles are encompassed within the sampling window. As it is not practical in a real time measurement environment to infinitely vary the sample rate, it is almost impossible to ensure an integer number of cycles are encompassed by the sampling window.

Important note : If an exact integer number of cycles were able to be encompassed within the data acquisition window of the FFT then this algorithm would provide similar accuracies to the DFT. The restriction of 2^n samples and fixed/discrete sample rates means this is impractical and the FFT cannot offer the same flexibility and accuracy in a real time measurement environment. The result of incomplete cycles within the sampling window is known as "harmonic leakage", in which harmonics "leak" into adjacent harmonic neighbours, causing inaccuracies in the calculated results.

In the analysis below, a 1024 point FFT was sampled at 10ks/s upon a 100Vpk, 25Hz waveform. Due to the nature of the FFT and its restriction to 2^n samples, at this sample rate (which provides 5kHz measurement bandwidth - Nyquist's theorem) it is clear that the data acquisition window does not encompass an integer number of samples, the additional undesirable samples are shown in the red box in fig 4.

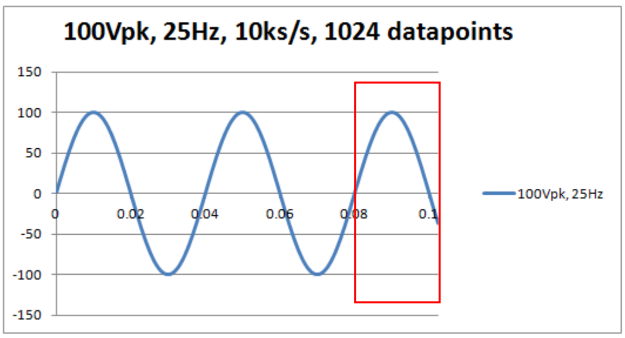

Fig 4.

If an FFT harmonic spectrum analysis is performed, significant harmonic leakage is evident as a result of the non-integral cycle acquisition.

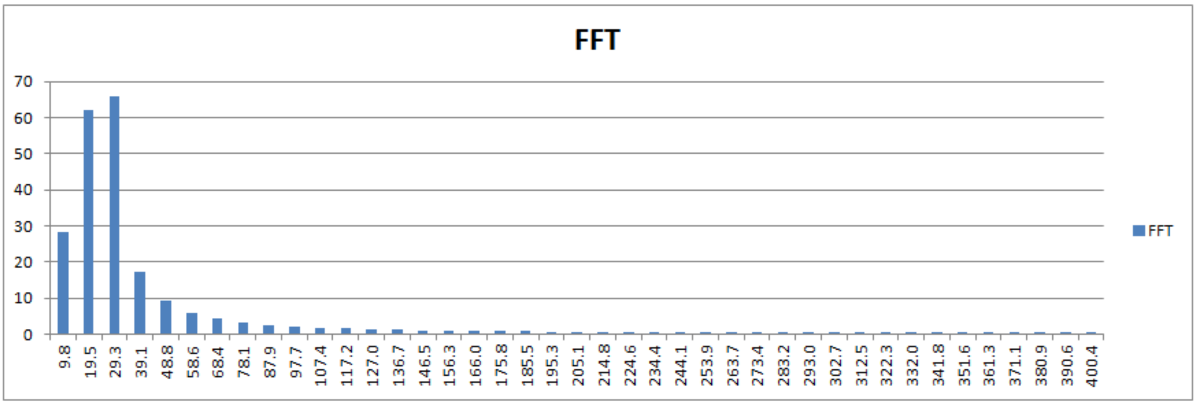

Fig 5.

Various complex waveforms are now analyzed and results from FFT and DFT algorithms are compared.

If the data acquisition window encompasses an integer number of cycles, the harmonic leakage is significantly reduced. Although this is unrealistic in a dynamic, real time measurement environment this is performed for clarity of the mathematics in fig 6 and fig 7.

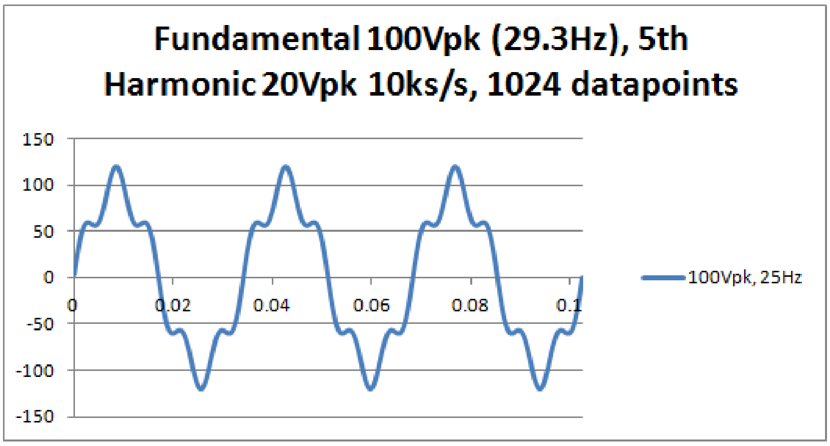

Fig 6.

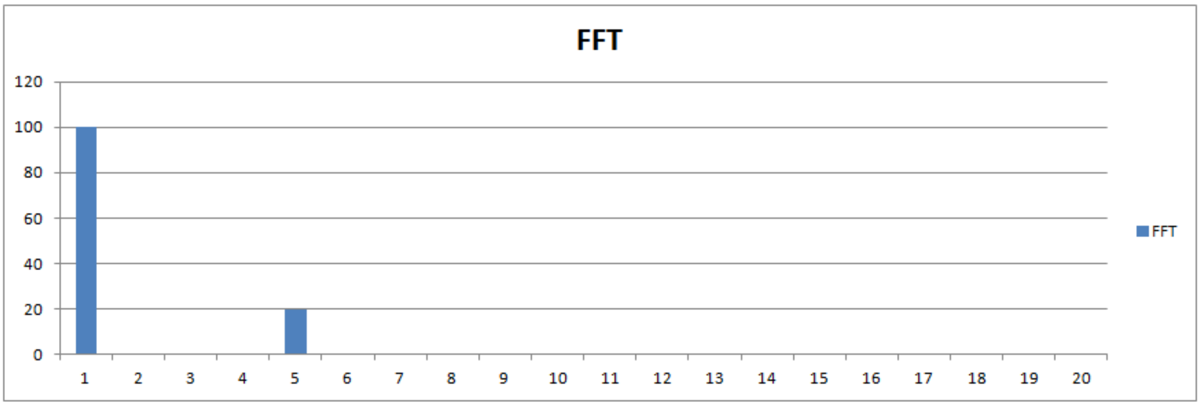

Fig 7. FFT Analysis with ideal window size

If the frequency detection or sample rate limitation (and subsequent window size) drifts by only 3~4Hz, the errors introduced into the FFT are quite significant. The graphs below illustrate this effect;

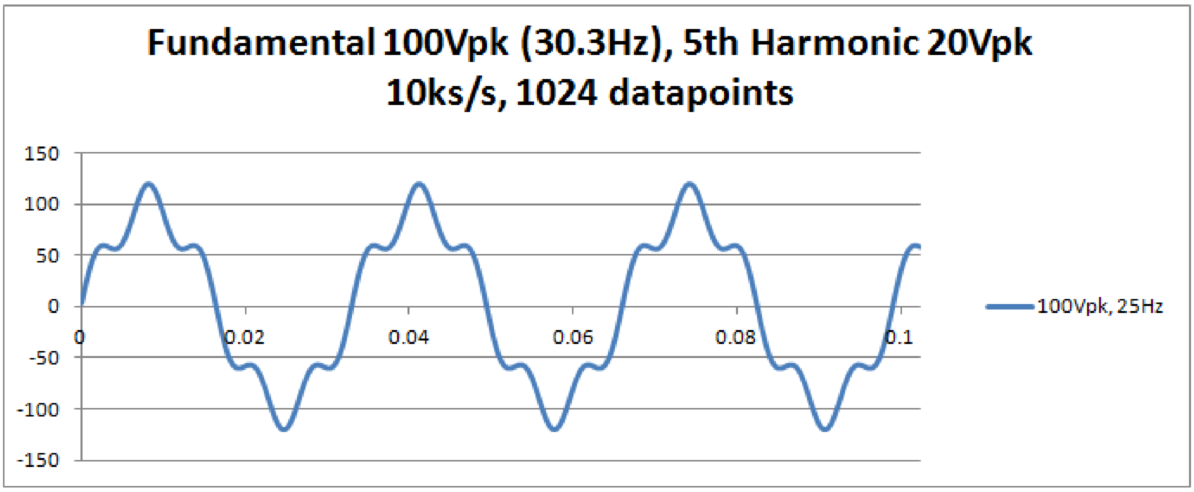

Fig 8.

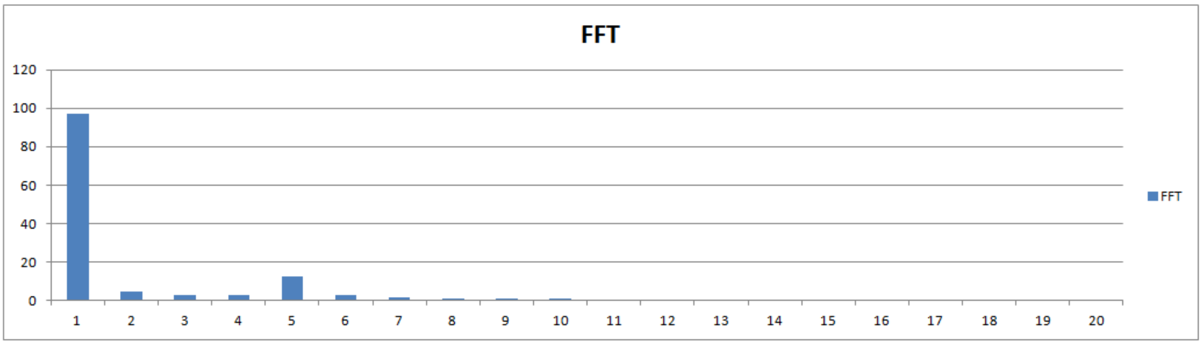

Fig 9.

Harmonic 1 (Fundamental) is significantly lower than the 100Vpk original component, there is also significant leakage into other harmonics which in reality have no spectral voltage components.

This is caused by the fact that the "sine and cosine" multiplication of the FFT is not performed at the correct harmonic frequencies, the fundamental component is in fact a 29.3Hz sine and cosine multiplication. The FFT frequencies are derived from the sample rate and the window size.

$$FFT\ Frequency = n * \left(\frac{fs}{sa}\right)$$

$n = sample\ frequency\ number\ 1,2,3 \ldots 1024,\quad fs = sample\ rate,\quad sa = number\ of\ samples$

The FFT frequency denotes the actual sine cosine calculation, if the sample frequency is not infinitely adjustable the FFT frequencies will be close but not completely matching the actual harmonic frequencies.

Error in FFT calculation

Harmonic	Frequency	Actual (Vpk)	Calculated with FFT	Error %
Fundamental	30.3Hz	100	97.25	2.75
2	60.6Hz	0	4.38	
3	90.9Hz	0	2.65	
4	121.2Hz	0	2.64	
5	151.5Hz	20	12.68	36.59
6	181.8Hz	0	2.57	

Table 1.

The further the fundamental frequencies drift from the FFT frequency, the greater the leakage and greater the error in the measured harmonics.

To Illustrate this effect, a fundamental frequency of 31.3Hz (equating to a drift of 2Hz) results in the following spectral analysis and subsequent errors.

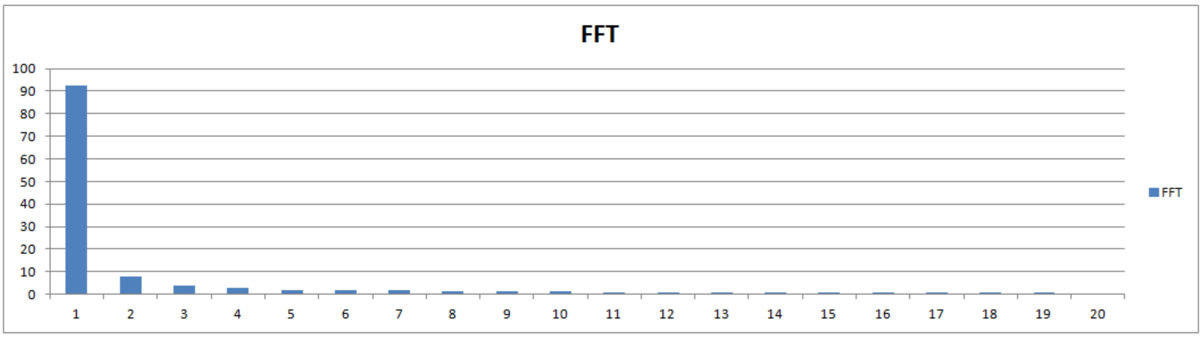

Fig 10.

Harmonic	Frequency	Actual (Vpk)	Calculated with FFT	Error %
Fundamental	30.3Hz	100	92.49	7.51
2	60.6Hz	0	7.62	
3	90.9Hz	0	3.95	
4	121.2Hz	0	2.67	
5	151.5Hz	20	1.67	91.67
6	181.8Hz	0	2.03	

Table 2.

As previously mentioned, it is impractical to achieve an exact integer number of waveform periods to fit within the data acquisition window when using the FFT in a dynamic environment. It is therefore not desirable to utilise the FFT Fourier analysis technique in modern day power analysis. There are windowing methods available to the engineer which will reduce the spectral leakage, these algorithms will improve values at some frequencies but also worsen values at other frequencies. Ultimately, none of the windowing algorithms available when applied to an FFT will match the accuracy of the DFT.

DFT (Discrete Fourier Transform)

The Discrete Fourier transform is able to resolve its sampling window to any integer number of samples, this equates to;

$$window\ resolution = \frac{1}{fs}$$

In the time domain, this equates to a time resolution of 100uS for a 10ks/s sample rate.

If a 50Hz waveform is considered, its time period is 20ms. A 100us time resolution enables the window to perfectly encompass both a 50Hz and 50.25Hz, it is logical to conclude that a higher sample rate will result in greater frequency resolution as well as higher bandwidth, if multiple cycles are used this resolution is further improved.

Newtons4th power analyzers feature sample rates of 1Ms/s and above, coupled with very accurate analogue performance, this provides extremely sensitive and accurate harmonic analysis with fast update rates.

Newtons4th Frequency resolution on a 50Hz waveform

$$N4L\ PPA5500\ window\ resolution(cycle\ by\ cycle) = \frac{1}{2Ms/s} = \frac{1}{2*10^6} = 500nS$$

$$N4L\ PPA5500\ cycle\ by\ cycle\ frequency\ resolution\ on\ 50Hz\ waveform = 50 - \left(\frac{1}{0.02 + 500*10^{-9}}\right)$$

$$= 50 - 49.99875 = 0.001Hz$$

The same waveform illustrated in fig.10 was used and a DFT was performed, for an appropriate comparison the same 10ks/s sample rate was used.

When utilising a DFT waveform, it is possible to synchronise the data acquisition window to the fundamental time period with a resolution of 1 sample point. Once this is performed, the DFT is able to calculate the harmonic components of the waveform very accurately. In this example the fundamental time period is calculated as follows;

$$Fundamental\ Time\ Period = \frac{1}{f_{fund}} = \frac{1}{31.3} = 0.0319\ seconds$$

The number of samples required to synchronise the data acquisition window to the fundamental time period is calculated as follows;

$$No\ of\ samples\ required = \frac{fund\ time\ period}{sample\ time\ interval} = \frac{0.0319}{\frac{1}{10000}} = 319.489$$

The closest integer number of samples calculated is used as the data window acquisition size, N4L power analyzers compute the fundamental frequency "real-time" with a proprietary frequency detection algorithm within a dedicated DSP.

It should be apparent that for the DFT to be successful, frequency detection is of paramount importance, as all N4L power analyzers utilise sample rates in excess of 1Ms/s, it is clear that the window and subsequent frequency resolution will be very accurate.

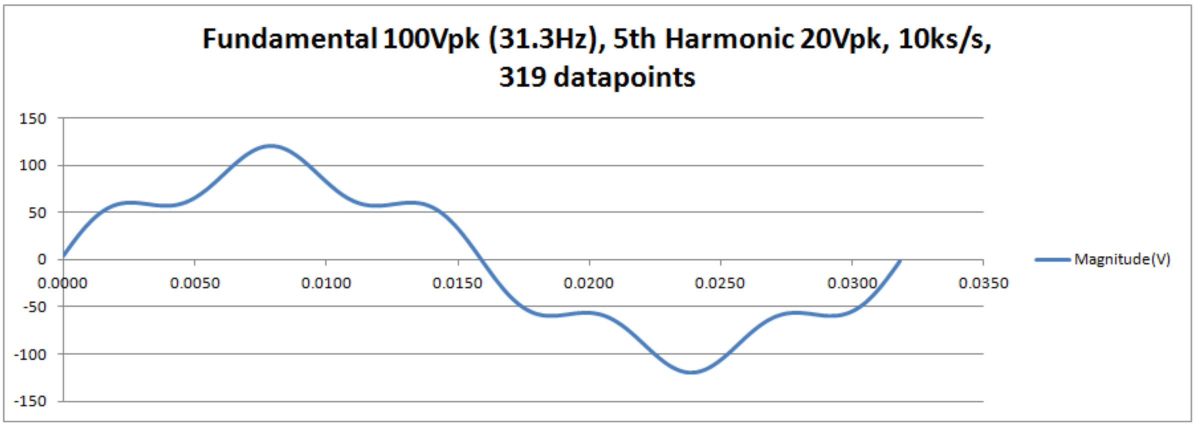

Fig 11.

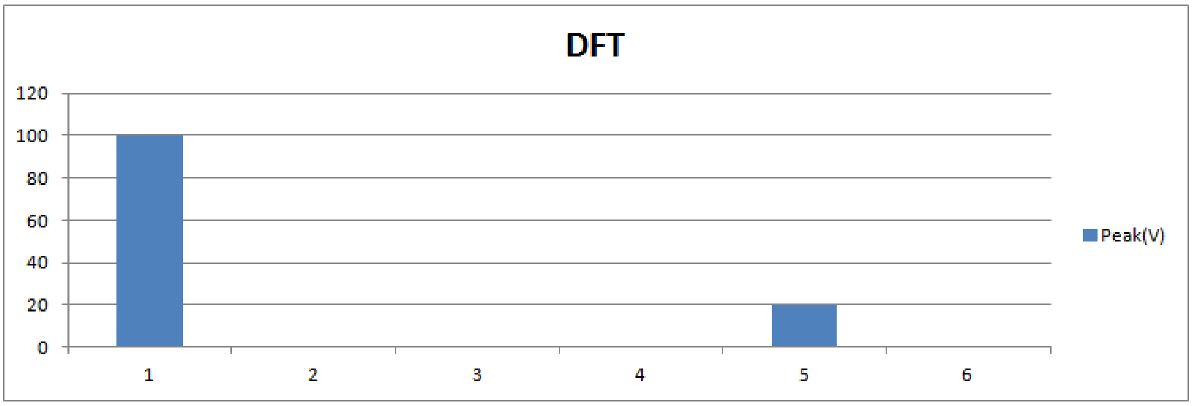

Fig 12.

Figure 11 and Figure 12 illustrate the effectiveness of the DFT even when utilising low sample rates and a single cycle. Higher sample rates and multiple cycle analysis will achieve greater accuracies through better window synchronisation (providing frequency detection is accurate) as well as greater averaging.

Harmonic	Frequency	Actual (Vpk)	Calculated with DFT	Error %
1	31.300	100.000	100.063	0.063
2	62.600	0.000	0.175	
3	93.900	0.000	0.057	
4	125.200	0.000	0.056	
5	156.500	20.000	20.077	
6	187.800	0.000	0.447	0.447

Table 3.

The calculated error achieved with the DFT is significantly better than the equivalent FFT, further benefits of the DFT include the flexibility of the data acquisition window providing cycle by cycle analysis of any waveform without the restrictions of 2^n FFT windowing.

Reference Measurements

An N4L PPA5500 Precision Power Analyzer, which utilises a DFT Harmonic Analysis algorithm was bench tested against the Fluke 6105A power standard. The tests were carried out within the UKAS ISO17025 Test Laboratory (Lab no. 7949) based at N4L Headquarters in the UK, results from the tests are shown below;

Fluke 6015A Calibration of PPA5530 Power Analyzer - No Adjustment					
Harmonic	Frequency	Applied (Vpk)	Measured	Error	Uncertainty
1	31.3	100	99.977	0.02%	0.01%
2	62.6	0	<10mV	NA	NA
3	93.9	0	<10mV	NA	NA
4	125.2	0	<10mV	NA	NA
5	156.5	20	19.994	0.03%	0.03%
6	187.8	0	<10mV	NA	NA

The errors above represent the total error of the measurement instrument, including the error of the analogue input channels. This calibration procedure demonstrates the power of the DFT when combined with extremely linear analogue input design.

Summary

Whilst the FFT has its place within the electrical engineering field, for high accuracy power analysis the FFT is not the optimum solution. A high end power analyzer will need to meet the demands of modern industry in which harmonic distortion of varying fundamental frequencies is required to be measured.

It is the responsibility of the power analyzer manufacturer to integrate sufficiently powerful processors and innovative digital signal processing techniques in order to meet the computational demands of the DFT.

All N4L power analyzers have been developed with this approach in mind and years of experience fine tuning both analogue hardware performance and DFT signal processing algorithms have resulted in an extremely accurate solution under a wide range of input waveforms.

For more information about any of the Newtons4th Power Analyzers, visit
http://www.newtons4th.com/products/power-analyzers/

Author : Sales and Applications Engineering, Newtons4th Ltd, UK

References

[1] J. Bird, Higher Engineering Mathematics - Fifth Edition, Elsevier, 2006